

\mathbb{CP}^2 and \mathbb{CP}^1 Sigma Models in Supergravity: Bianchi type IX Instantons and Cosmologies

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Abstract

We find instanton/cosmological solutions with biaxial Bianchi-IX symmetry, involving non-trivial spatial dependence of the \mathbb{CP}^1 - and \mathbb{CP}^2 -sigma-models coupled to gravity. Such manifolds arise in $N = 1$, $d = 4$ supergravity with supermatter actions and hence the solutions can be embedded in supergravity. There is a natural way in which the standard coordinates of these manifolds can be mapped into the four-dimensional physical space. Due to its special symmetry, we start with \mathbb{CP}^2 with its corresponding scalar Ansatz; this further requires the spacetime to be $SU(2) \times U(1)$ -invariant. The problem then reduces to a set of ordinary differential equations whose analytical properties and solutions are discussed. Among the solutions there is a surprising, special-family of exact solutions which owe their existence to the non-trivial topology of \mathbb{CP}^2 and are in 1-1 correspondence with matter-free Bianchi-IX metrics. These solutions can also be found by coupling \mathbb{CP}^1 to gravity. The regularity of these Euclidean solutions is discussed – the only possibility is bolt-type regularity. The Lorentzian solutions with similar scalar Ansatz are all obtainable from the Euclidean solutions by Wick rotation.

1 Introduction

Because of their non-linearities exact solutions of the Einstein equations, in vacuo or in the presence of a possible cosmological constant, are usually obtained assuming a large degree of symmetry and other simplifying features. In the presence of matter fields, especially non-linear ones, exact

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solutions are much rarer and are known only in the simplest matter configurations. These solutions however are the starting points for the full dynamical study of the gravity-matter system.

Largely because of their possible role in inflationary cosmology, cosmological solutions with scalars coupled to gravity have garnered particular interest in both Lorentzian and Riemannian signatures. In supergravity theories scalar fields arise naturally as sigma models and hence the resulting system is more non-linear and difficult to solve. In most work on the subject, however, authors tend to ignore the geometry of the ‘target’ manifold on which the scalar fields live, with the consequence that it does not play a role in determining the geometry of the spacetime and makes the system simpler; attention has usually focussed much more on the form of the scalar potential. In this paper, we will see that information of a different kind emerges if we let the target manifold play a more active role.

For definiteness we consider the \mathbb{CP}^1 - and \mathbb{CP}^2 -sigma-models coupled to gravity with a cosmological constant. Such models can be embedded, for example, in $N = 1$, $d = 4$ supergravity with supermatter Lagrangian (see below). In fact, they arise naturally for the gauge group of $SU(3)$ and $SU(2)$ in $N = 1$, $d = 4$ supergravity coupled to gauged supermatter Lagrangian. For reasons of symmetry involving the cosmological models, we first consider \mathbb{CP}^2 . With a non-trivial, but natural, Ansatz for the scalars, the symmetry of spacetime naturally emerges to be that of biaxial Bianchi-IX. We derive the general set of field equations and obtain, among others, a set of special-case exact solutions which are “deformations” of the Bianchi-IX biaxial metrics satisfying the Einstein- Λ equations and are in 1-1 correspondence with them. These solutions exist for both the Lorentzian and Euclidean signatures. The Lorentzian solutions can all be obtained by Wick rotation and hence we will be describing the Euclidean solutions and their regularity in greater length. Some of the Euclidean solutions can be extended over complete Riemannian manifolds and hence are instanton solutions of the coupled system. We then show that this special set of solutions can be obtained from the \mathbb{CP}^1 sigma models. All solutions that we will be reporting in this paper make crucial use of the geometry of the target manifold.

This paper is arranged in the following way. In section 2 we discuss the theoretical framework and the action; in section 3 we describe the Ansatz for the scalar fields for the \mathbb{CP}^2 -sigma model and the spacetime and deduce the field equations. In section 4 we describe the solutions. In section 5 we discuss the regularity of the solutions and show how these solutions can in fact be obtained for the \mathbb{CP}^1 -sigma model as well. Finally, we conclude with comments on the corresponding Lorentzian solutions.

2 Actions

In this paper we will essentially find solutions corresponding to the following (two) Euclidean actions:

$$I_E = - \int d^4x \sqrt{g} \left(\frac{1}{2} R - g_{ij*} \partial_\mu a^i \partial^\mu a^{*j} - \Lambda \right) \quad (2.1)$$

where $g_{\mu\nu}$ is the spacetime metric, with $g = \det(g_{\mu\nu})$, and g_{ij*} is the metric on the target (complex) manifold, here the complex projective spaces \mathbb{CP}^2 and \mathbb{CP}^1 respectively, and Λ is an arbitrary cosmological constant. The motivation for studying such sigma models comes from supergravity as scalar fields in supergravity theories take their values on Kähler manifolds.

2.1 $N = 1$, $d = 4$ supergravity with supermatter action

It is easy to check that the action (2.1) is a valid truncated of the full $N = 1$, $d = 4$ supergravity action whose bosonic part is [16]

$$\mathcal{I} = - \int d^4x \sqrt{g} \left(\frac{1}{2} R - g_{ij*} \partial_\mu a^i \partial^\mu a^{*j} - \frac{1}{2} D_a D^a - \frac{1}{4} \text{Re}(h_{ab}) F_{\rho\sigma}^{(a)} F^{(a)\rho\sigma} + \frac{1}{4} \text{Im}(h_{ab}) F_{\rho\sigma}^{(a)} F^{(a)\rho\sigma} - V_F(a^i, a^{*j}) \right). \quad (2.2)$$

Here $g_{ij*} = \partial^2 K / \partial a^i \partial a^{j*}$ is the Kähler metric which is derived from a Kähler potential $K(a^i, a^{*j})$ and $F_{\mu\nu}^{(a)} = \partial_\mu v_\nu^{(a)} - \partial_\nu v_\mu^{(a)}$ are the Maxwell field strengths (with $U(1)$ gauge group for each index (a)) and

$$V_F = e^K \left(g^{ij*} D_i W D_{j*} W - 3|W|^2 \right), \quad (2.3)$$

where W , the superpotential, is a holomorphic function of a_i and $D_i W \equiv \partial_i W + \partial_i K W$ and D_a are constants corresponding to a Fayet-Iliopoulos term. Assuming $h_{ab} = \pm \delta_{ab}$, $v_\mu^{(a)} = 0$ and $W = 0$, action (2.1) can be obtained as a valid truncation of (2.2). The validity can be checked trivially by observing that the equations of motion of (2.2) reduce to those of (2.1) with a cosmological constant term $\frac{1}{2} D_a D^a$. Therefore all solutions to be described in this paper can be seen as solutions of (2.2).

2.2 $N = 1$, $d = 4$ supergravity with gauged supermatter

We have already remarked that \mathbb{CP}^2 and \mathbb{CP}^1 scalar manifolds have isometry group of $SU(3)$ and $SU(2)$, and arise in the $N = 1$, $d = 4$ supergravity with gauged supermatter action for these two gauge groups. It is therefore natural to ask whether (2.1) can be obtained from $N = 1$, $d = 4$ supergravity with gauged supermatter action. The bosonic part of the latter is¹ [16]

$$\mathcal{I} = - \int d^4x \sqrt{g} \left(\frac{1}{2} R - g_{ij*} \tilde{D}_\mu a^i \tilde{D}^\mu a^{*j} - \frac{1}{2} g^2 D^{(a)2} - \frac{1}{4} F_{\mu\nu}^{(a)} F^{\mu\nu(a)} - V_F(a^i, a^{*j}) \right). \quad (2.4)$$

Here $\tilde{D}_\mu a^i$ denotes $(\partial_\mu a^{(i)} - g v_\mu^{(a)} X^{i(a)})$, where $v_\mu^{(a)}$ is the multiplet of Yang-Mills potentials ($a = 1, 2, 3$ for $SU(2)$ and $a = 1, \dots, 8$ for $SU(3)$) and $F_{\mu\nu}^{(a)}$ are the Yang-Mills field strengths. The quantities $X^{i(a)}$, together with their complex conjugates $X^{*j(a)}$, give the holomorphic Killing vector fields

$$\begin{aligned} X^{(b)} &= X^{i(b)}(a) \frac{\partial}{\partial a^i}, \\ X^{*(b)} &= X^{*i(b)}(a^*) \frac{\partial}{\partial a^{*i}}, \end{aligned} \quad (2.5)$$

corresponding to the infinitesimal isometries of the Kähler manifold. The term $D^{(a)2}$ gives a cosmological constant $g^2/8$ for both gauge groups. The connection between the Kähler scalar part of the theory and the gauge theory is that the isometry group of the scalars is the gauge group of the full theory.

Obviously to obtain (2.1) from (2.4), we need the Yang-Mills potentials as well as $V_F(a^i, a^{*j})$ vanishing. The latter is achieved by setting the superpotential to zero. However, if one sets $v_\mu^{(a)} = 0$, it is not difficult to see that the equations of motion for (2.4) would not reduce to those of (2.1). This is because the Yang-Mills fields are coupled to the scalar fields and the corresponding equation of motion

$$D^\mu \left(\sqrt{g} F_{\mu\nu}^{(a)} \right) = g g_{ij*} \left(X^{(a)i} \tilde{D}_\mu a^{*j} - \tilde{D}_\mu a^i X^{*j(a)} \right) \quad (2.6)$$

¹Here we have made a choice for h_{ab} , which does not affect the ensuing arguments.

would not be satisfied for complex scalar fields in general. For this to be possible one needs to set $g = 0$, which takes us back to action (2.2).

In the following we adopt the conventions of [13], except that we take $8\pi G = 1$. The Einstein equations are then

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}. \quad (2.7)$$

3 \mathbb{CP}^2 : Field Equations and Solutions

3.1 Geometry of \mathbb{CP}^2

\mathbb{CP}^2 is most simply described in terms of two complex scalar coordinates, a^1 and a^2 , such that the Hermitian metric g_{ij}^* is derived from the Kähler potential, $K = \log(1 + a^1 a^{*1} + a^2 a^{*2})$, as

$$g_{ij}^* = \partial^2 K / \partial a^i \partial a^{*j} \quad (3.1)$$

As is well known (see for example, [10]) \mathbb{CP}^2 can also be described in terms of four real coordinates (R, Θ, Ψ, Φ) :

$$a^1 = R \cos \frac{\Theta}{2} \exp \left(i \frac{\Psi + \Phi}{2} \right) \quad (3.2)$$

and

$$a^2 = R \sin \frac{\Theta}{2} \exp \left(i \frac{\Psi - \Phi}{2} \right), \quad (3.3)$$

where

$$\begin{aligned} 0 &\leq R \leq \infty \\ 0 &\leq \Theta \leq \pi \\ 0 &\leq \Phi \leq 2\pi \\ 0 &\leq \Psi \leq 4\pi, \end{aligned} \quad (3.4)$$

giving the real Fubini-Study metric:

$$ds^2 = \frac{dR^2}{(1 + \frac{\mu}{6}R^2)^2} + \frac{R^2}{4(1 + \frac{\mu}{6}R^2)}(\sigma_1^2 + \sigma_2^2) + \frac{R^2}{4(1 + \frac{\mu}{6}R^2)^2}(\sigma_3^2) \quad (3.5)$$

where σ_i are the left-invariant one forms on $SU(2)$ (equivalently, on S^3):

$$\begin{aligned} \sigma_1 &= \cos \Psi d\Theta + \sin \Theta \sin \Psi d\Phi, \\ \sigma_2 &= -\sin \Psi d\Theta + \sin \Theta \cos \Psi d\Phi, \\ \sigma_3 &= \cos \Theta d\Phi + d\Psi \end{aligned} \quad (3.6)$$

and obey the exterior algebra $d\sigma_i = -\frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k$. The quantity μ is a positive constant and is *not* determined by the model.

One can check that metric (3.5) is Einstein, i.e., that it satisfies the Einstein equations with a (positive) cosmological constant ($R_{\mu\nu} = \mu g_{\mu\nu}$) and hence \mathbb{CP}^2 is a compact manifold. This can also be verified explicitly by making the coordinate transformation $R = \sqrt{\frac{6}{\mu}} \tan \chi$ ($0 \leq \chi \leq \frac{1}{2}\pi$):

$$ds^2 = \frac{6}{\mu} \left(d\chi^2 + \frac{1}{4} \sin^2 \chi (\sigma_1^2 + \sigma_2^2) + \sin^2 \chi \cos^2 \chi (\sigma_3^2) \right). \quad (3.7)$$

The constant- R surfaces of (3.5) are non-trivial S^1 bundles over S^2 being invariant under the group action of $SU(2) \times U(1)$. (This, however, is not the maximal symmetry of $\mathbb{C}P^2$.) Since the periodicity of Ψ -coordinate is 4π , they are topologically S^3 . Near $R = 0$ the metric approaches flat space and near $R = \infty$ it collapses to a 2-sphere of finite radius. These refer to the “nut” and “bolt” of the metric – terms which will be made clearer later in the paper. For the purpose of much of the discussions below, we will conveniently set $\mu = 6$ in (3.5) unless this results in loss of generality. We will revert to the general form (3.5) where appropriate, as in the principal set of explicit solutions obtained in this paper.

3.2 Metric Ansatz for \mathcal{M}^4 and Field Equations

For the metric on the Riemannian ‘spacetime’ manifold $(\mathcal{M}^4, g_{\mu\nu})$, we take the following Ansatz relating the coordinates of ‘spacetime’ (here r, θ, ψ, ϕ) with those of the target manifold:

$$R(x^\mu) = R(r), \quad (3.8)$$

$$\Theta(x^\mu) = \theta, \Psi(x^\mu) = \psi, \Phi(x^\mu) = \phi.$$

One finds that:

$$\begin{aligned} 2g_{ij*} \partial_r a^i \partial_r a^{*j} &= \frac{2R'^2}{(1+R^2)^2}, \\ 2g_{ij*} \partial_\theta a^i \partial_\theta a^{*j} &= \frac{1}{2} \frac{R^2}{(1+R^2)}, \\ 2g_{ij*} \partial_\psi a^i \partial_\psi a^{*j} &= \frac{1}{2} \frac{R^2}{(1+R^2)^2}, \\ 2g_{ij*} \partial_{(\psi} a^i \partial_{\phi)} a^{*j} &= \frac{1}{2} \frac{R^2}{(1+R^2)^2} \cos \theta, \\ 2g_{ij*} \partial_\phi a^i \partial_\phi a^{*j} &= \frac{1}{2} \frac{R^2}{(1+R^2)^2} (1 + R^2 \sin^2 \theta) \end{aligned} \quad (3.9)$$

while all other components of $2g_{ij*} \partial_{(\mu} a^i \partial_{\nu)} a^{*j}$ are zero. This naturally suggests the metric Ansatz for \mathcal{M}^4 , of Riemannian biaxial Bianchi-IX type:

$$ds^2 = dr^2 + a^2(r)[(\sigma_1)^2 + (\sigma_2)^2] + b^2(r)(\sigma_3)^2. \quad (3.10)$$

Hence the non-zero components of $T_{\mu\nu}$ are:

$$\begin{aligned} T_{rr} &= \frac{R'^2}{(1+R^2)^2} - \left(\frac{R^2}{2(1+R^2)a^2} + \frac{R^2}{4(1+R^2)^2 b^2} \right), \\ T_{\theta\theta} &= - \left(\frac{R'^2}{(1+R^2)^2} + \frac{R^2}{4(1+R^2)^2 b^2} \right) a^2, \\ T_{\psi\psi} &= - \left(\frac{R'^2}{(1+R^2)^2} + \frac{R^2}{2(1+R^2)a^2} - \frac{R^2}{4(1+R^2)^2 b^2} \right) b^2, \\ T_{\psi\phi} &= T_{\phi\psi} = T_{\psi\psi} \cos \theta, \\ T_{\phi\phi} &= T_{\theta\theta} \sin^2 \theta + T_{\psi\psi} \cos^2 \theta. \end{aligned} \quad (3.11)$$

Scalar field equations

Non-linear sigma models are special cases of harmonic maps from the spacetime to the target manifold (see, for example, [6]). Harmonic maps are governed by the equations:

$$\nabla^\mu \nabla_\mu X^A + \Gamma_{BC}^A \nabla^\mu X^B \nabla_\mu X^C = 0, \quad (3.12)$$

where the X^A and Γ_{BC}^A are respectively the coordinates and Christoffel symbols of the target manifold. For the present case, it is convenient to make use of the fact that \mathbb{CP}^2 can be given four real coordinates (R, Θ, Ψ, Φ) and a real metric (3.5). Instead of finding (complex) field equations for a^i and a^{*j} by varying the action (2.1), and then translating them into real coordinates, we simply use (3.12) to find the field equations for the real coordinates (R, Θ, Ψ, Φ) of the Fubini-Study metric. By the Ansatz (3.8), angular coordinates are determined by the corresponding ‘spacetime’ coordinates. The only non-trivial equation is for R : on using the identity $\nabla^\mu \nabla_\mu = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} \partial^\mu)$, we find the equation for $R(r)$ (Eq.(3.14) below).

The whole classical problem now reduces to that of finding solutions to the following set of equations:

$$\begin{aligned} \left(\frac{a'}{a}\right)^2 + 2\frac{a'}{a}\frac{b'}{b} + \frac{1}{4}\frac{b'^2}{a^4} - \frac{1}{a^2} &= \frac{R'^2}{(1+R^2)^2} - \left(\frac{R^2}{2a^2(1+R^2)} + \frac{R^2}{4b^2(1+R^2)^2}\right) - \Lambda, \\ \frac{a''}{a} - \frac{a'}{a}\frac{b'}{b} + \frac{1}{4}\frac{b^2}{a^4} &= -\frac{R'^2}{(1+R^2)^2} + \frac{R^2}{4b^2(1+R^2)^2}, \\ \frac{b''}{b} + 2\frac{a'}{a}\frac{b'}{b} - \frac{1}{2}\frac{b^2}{a^4} &= -\frac{R^2}{2b^2(1+R^2)^2} - \Lambda, \end{aligned} \quad (3.13)$$

with

$$R'' + \left(2\frac{a'}{a} + \frac{b'}{b}\right) R' - \frac{2RR'^2}{(1+R^2)} - \frac{R}{2a^2} + \frac{R(R^2-1)}{4(1+R^2)b^2} = 0. \quad (3.14)$$

The first-order constraint equation is consistent with the three other second-order equations, as one can check. A typical solution $(a(r), b(r), R(r))$ would involve numerical integration. One can check that the full system of equations admits consistent (regular) power-series solutions for $(a(r), b(r), R(r))$ near $r = 0$. This can be used as the starting-point in finding numerical solutions – to be studied elsewhere. In this paper we study some analytic solutions.

4 Solutions

4.1 Einstein Metrics

One may try to identify R with r . This is similar to the approach in [7] where the scalar coupling was initially taken to be arbitrary, and \mathbb{CP}^2 was found to be a solution for \mathcal{M}^4 for a particular value of the scalar coupling, leading to the concept of spontaneous scalar compactification [8, 14] – a concept which has since been used to compactify higher dimensional spacetimes in various higher-dimensional theories.

However, in our case identifying R with r means that we have to solve:

$$R_{\mu\nu \mathcal{M}^4} = 2g_{\mu\nu \mathbb{CP}^2} + \Lambda g_{\mu\nu \mathcal{M}^4}. \quad (4.1)$$

For $\Lambda = 0$, it is not difficult to see that Eq.(4.1) cannot be satisfied by any Einstein metric, as follows. For an Einstein space $R_{\mu\nu} = \lambda g_{\mu\nu}$ (λ is a constant), which in our case would mean $g_{\mu\nu \mathcal{M}^4} = \frac{2}{\lambda} g_{\mu\nu \mathbb{CP}^2}$, implying that the two metrics are related by a constant conformal factor. This would imply that the Ricci tensors are equal: $R_{\mu\nu \mathcal{M}^4} = R_{\mu\nu \mathbb{CP}^2}$. This gives $R_{\mu\nu \mathbb{CP}^2} = 2g_{\mu\nu \mathbb{CP}^2}$ – a contradiction, since in this case $R_{\mu\nu \mathbb{CP}^2} = 6g_{\mu\nu \mathbb{CP}^2}$. However, this does not exhaust the possibilities as we will see below.

First note that for the special value of the cosmological constant $\Lambda = 4$, we can recover the \mathbb{CP}^2 as the solution for \mathcal{M}^4 . This generalizes directly. As remarked earlier, the μ for \mathbb{CP}^2 is not determined by the theory. Identifying R with r , the generalization of (4.1) is:

$$R_{\mu\nu \mathcal{M}^4} = \frac{12}{\mu} g_{\mu\nu \mathbb{CP}^2} + \Lambda g_{\mu\nu \mathcal{M}^4}. \quad (4.2)$$

This can be solved for μ , and hence the corresponding Fubini-Study metric satisfying (21) can be found. In other words, it is the reverse process: we give the correct metric on \mathbb{CP}^2 to get the same metric on \mathcal{M}^4 , i.e., we solve

$$\frac{12}{\mu} + \Lambda = \mu \quad (4.3)$$

for μ . This gives $\mu = \frac{1}{2}(\Lambda + \sqrt{\Lambda^2 + 48})$, for both positive and negative Λ ; hence the corresponding Fubini-Study metric on \mathcal{M}^4 can be found. For $\Lambda = 0$ this gives $\mu = 2\sqrt{3}$.

4.2 $R = \text{constant}$ solutions ?

Clearly, in the case $R(r) = 0$, the field equations reduce to those of a biaxial Bianchi-IX model (without matter) admitting a $SU(2) \times U(1)$ isometry group. The solutions are the general two-parameter Riemannian Taub-NUT-(anti-)de Sitter family of metrics [11]:

$$ds^2 = \frac{\rho^2 - L^2}{\Delta} d\rho^2 + \frac{4L^2 \Delta}{\rho^2 - L^2} (d\psi + \cos \theta d\phi)^2 + (\rho^2 - L^2)(d\theta^2 + \sin^2 \theta d\phi^2), \quad (4.4)$$

where

$$\Delta = \rho^2 - 2M\rho + L^2 + \Lambda(L^4 + 2L^2\rho^2 - \frac{1}{3}\rho^4). \quad (4.5)$$

This general form, however, is only valid for a coordinate patch for which $\Delta \neq 0$. At the roots, the metric degenerates to the two-dimensional fixed-point set of the Killing vector field $\partial/\partial\psi$; they are round 2-spheres of constant radii and have been dubbed “bolts” [9]. However, if a root occurs at $\rho = |L|$, the corresponding set of fixed points is zero-dimensional, as the 2-sphere then collapses to a point; such a point is called a “nut” [9]. Such nuts and bolts are not necessarily regular points of the metric. For them to be regular, the metric has to close smoothly. This will be discussed in Section 5.1 in greater detail. In general, one arrives at two one-parameter family of metrics, the self-dual Taub-Nut-(anti-)de Sitter and the Taub-Bolt-(anti-)de Sitter metrics, by imposing the condition of regularity. Known examples of Bianchi-IX metrics and instantons arise as special cases of them. For positive cosmological constant, known examples include the usual round metric on S^4 , the Fubini-Study metric on \mathbb{CP}^2 [10]. For vanishing cosmological constant, the known solutions are the self-dual Taub-Nut instanton [12], the Taub-Bolt instanton [15]. For negative cosmological

constant, one special solution is the Bergman metric on $\overline{\mathbb{CP}^2}$, which is just the Fubini-Study metric with the cosmological constant reversed in sign².

Although no new metric solutions have been obtained in this rather trivial limit $R \equiv 0$, one can note that the scalar manifold is crucial in fixing the symmetry of the hypersurface of \mathcal{M}^4 at constant r to be at least $SU(2) \times U(1)$ -invariant, unlike the case in which one ignores the internal geometry and just takes a more *ad hoc* Ansatz for \mathcal{M}^4 . Thus, biaxial Bianchi-IX metrics arise naturally by virtue of the “hedgehog”-type Ansatz for the scalar fields living on the internal space. One might next ask whether any solutions exist for which $R(r)$ is a non-zero constant. In this case the scalar field equation reads:

$$\frac{R}{2a^2} - \frac{R(R^2 - 1)}{4(1 + R^2)b^2} = 0 \quad (4.6)$$

so that a and b are proportional:

$$a = \sqrt{2} \sqrt{\frac{R^2 + 1}{R^2 - 1}} b \quad (4.7)$$

with $R^2 > 1$. To examine the existence of such a solution, write the two evolution equations (3.13) in a slightly different way:

$$\begin{aligned} \frac{a''}{a} - \frac{a'}{a} \frac{b'}{b} &= -\frac{1}{4} \frac{b^2}{a^4} + \frac{R^2}{4b^2(1+R^2)^2}, \\ \frac{b''}{b} - \left(\frac{a'}{a}\right)^2 &= \frac{3}{4} \frac{b^2}{a^4} - \frac{1}{a^2} + \frac{R^2}{2a^2(1+R^2)} - \frac{R^2}{4b^2(1+R^2)^2}. \end{aligned} \quad (4.8)$$

For a and b proportional to each other, both right hand sides should be identical. But, on substituting (4.7), one finds that this requires $R = 3/5$, contradicting the requirement $R^2 > 1$. One might then have thought that there are no other geometrically significant solutions corresponding to a fixed value of R and solving the coupled system of equations (3.13) and (3.14), except for $R \equiv 0$. However, this is not so, as we have so far omitted the case ‘ $R \equiv \infty$ ’.

4.2.1 $R = \infty$

One may investigate the neighbourhood of $R \rightarrow \infty$ by defining $u(r) = \frac{1}{R(r)}$. The field equations become:

$$\begin{aligned} \left(\frac{a'}{a}\right)^2 + 2\frac{a'}{a} \frac{b'}{b} + \frac{1}{4} \frac{b^2}{a^4} - \frac{1}{a^2} &= \frac{u'^2}{(1+u^2)^2} - \left(\frac{1}{2(1+u^2)a^2} + \frac{u^2}{4b^2(1+u^2)^2}\right) - \Lambda, \\ \frac{a''}{a} - \frac{a'}{a} \frac{b'}{b} + \frac{1}{4} \frac{b^2}{a^4} &= -\frac{u'^2}{(1+u^2)^2} + \frac{u^2}{4b^2(1+u^2)^2}, \\ \frac{b''}{b} + 2\frac{a'}{a} \frac{b'}{b} - \frac{1}{2} \frac{b^2}{a^4} &= -\frac{u^2}{2b^2(1+u^2)^2} - \Lambda, \end{aligned} \quad (4.9)$$

with

$$u'' + \left(2\frac{a'}{a} + \frac{b'}{b}\right) u' - \frac{2uu'^2}{(1+u^2)} + \frac{u}{2a^2} - \frac{u(u^2 - 1)}{4(1+u^2)b^2} = 0. \quad (4.10)$$

²There are solutions, the Eguchi-Hanson metrics [5] for example, whose level-surfaces are not topologically S^3 . Since we have taken the Ansatz in which the ψ -coordinate of \mathcal{M}^4 has a period of 4π , as the Ψ -coordinate of the Fubini-Study metric (3.5) of the target manifold \mathbb{CP}^2 , such metrics are automatically precluded. For more discussions on such metrics, see [3].

It is clearly consistent to set $u = 0$ (corresponding to $R = \infty$). However, in contrast to the $R(r) = 0$ case, here ($R = \infty$) the energy-momentum tensor is non-zero. This is due to the fact that at $R = 0$, \mathbb{CP}^2 degenerates to a point (a “nut”) whereas at $R = \infty$ it degenerates to an S^2 of constant radius (a “bolt” – as in section 4). The three field equations read:

$$\left(\frac{a'}{a}\right)^2 + 2\frac{a'}{a}\frac{b'}{b} + \frac{1}{4}\frac{b^2}{a^4} - \frac{1}{a^2} = -\frac{1}{2a^2} - \Lambda, \quad (4.11)$$

$$\frac{a''}{a} - \frac{a'}{a}\frac{b'}{b} + \frac{1}{4}\frac{b^2}{a^4} = 0, \quad (4.12)$$

$$\frac{b''}{b} + 2\frac{a'}{a}\frac{b'}{b} - \frac{1}{2}\frac{b^2}{a^4} = -\Lambda. \quad (4.13)$$

These equations are just as in the biaxial Bianchi-IX case with only a cosmological constant (and no matter), except for the presence of the $\frac{1}{2a^2}$ term in (4.11). However, by the rescaling $\alpha(r) = \sqrt{2}a(r)$ and $\beta(r) = 2b(r)$, these equations reduce to those for the Bianchi-IX case with just a cosmological constant Λ :

$$\left(\frac{a'}{a}\right)^2 + 2\frac{a'}{a}\frac{b'}{b} + \frac{1}{4}\frac{b^2}{a^4} - \frac{1}{a^2} = -\frac{1}{2a^2} - \Lambda, \quad (4.14)$$

$$\frac{a''}{a} - \frac{a'}{a}\frac{b'}{b} + \frac{1}{4}\frac{b^2}{a^4} = 0, \quad (4.15)$$

$$\frac{b''}{b} + 2\frac{a'}{a}\frac{b'}{b} - \frac{1}{2}\frac{b^2}{a^4} = -\Lambda. \quad (4.16)$$

Therefore *all* our solutions with $R = \infty$ can be put into 1-1 correspondence with those of the Einstein- Λ system without matter. The two scale factors $a(r)$ and $b(r)$ are dilated by factors of $1/\sqrt{2}$ and $\frac{1}{2}$ when compared with the biaxial Bianchi IX solution with only a Λ term.

The general solutions of Bianchi-IX type, obeying the Einstein equations with a Λ term, are the Taub-NUT family of metrics (4.4) as discussed already. Therefore the solutions to the case ‘ $R = \infty$ ’ are given by the “extended” metrics:

$$ds^2 = \frac{\rho^2 - L^2}{\Delta} d\rho^2 + \frac{4L^2\Delta}{\rho^2 - L^2} \left(1 - \frac{3}{\mu}\right)^2 (d\psi + \cos\theta d\phi)^2 + (\rho^2 - L^2) \left(1 - \frac{3}{\mu}\right) (d\theta^2 + \sin^2\theta d\phi^2) \quad (4.17)$$

This is a two-parameter family of metrics which clearly are in 1-1 correspondence with their no-matter counterparts. We now discuss the regularity of these metrics, and then show how the same set of solutions is allowed for the \mathbb{CP}^1 sigma-model coupled to gravity.

5 New Metrics and their Regularity

It is convenient to rewrite the two-parameter family of metrics (4.17) in the form:

$$ds^2 = \frac{\zeta^2 - l^2}{\tilde{\Delta}(1 - \frac{3}{\mu})} d\zeta^2 + \frac{4l^2\tilde{\Delta}(1 - \frac{3}{\mu})}{\zeta^2 - l^2} (d\psi + \cos\theta d\phi)^2 + (\zeta^2 - l^2)(d\theta^2 + \sin^2\theta d\phi^2), \quad (5.1)$$

where

$$\tilde{\Delta} = \zeta^2 - 2m\rho + l^2 + \frac{\Lambda}{(1 - \frac{3}{\mu})} (l^4 + 2l^2\zeta^2 - \frac{1}{3}\zeta^4). \quad (5.2)$$

The lower-case quantities m, l are continuous parameters, related to those of (4.17) by $m = \sqrt{(1 - \frac{3}{\mu})} M$ and $l = \sqrt{(1 - \frac{3}{\mu})} L$. Again, this metric is only valid for a coordinate patch for which $\tilde{\Delta} \neq 0$ ($\tilde{\Delta}$ having four roots).

5.1 Regularity of the Taub-NUT-(anti-)de Sitter family

As already remarked in Section 4.2, the four roots of $\Delta = 0$ are not in general regular points of the metric (4.4). In this section we briefly describe how one obtains two one-parameter family of metrics from (4.4) by making one of the roots regular (for more details, see [1, 2, 3, 4]). The condition of regularity of (4.4) at any point ρ_{bolt} where $\Delta = 0$ works out to be [15]:

$$\frac{d}{d\rho} \left(\frac{\Delta}{\rho^2 - L^2} \right)_{(\rho=\rho_{root})} = \frac{1}{2L} \quad (5.3)$$

which amounts to imposing a relation between M and L . Thus the condition of regularity reduces the two-parameter Taub-NUT-(anti-)de Sitter family essentially to one-parameter families.

Self-dual Taub-Nut-(anti-)de Sitter

The metric (4.4) has a nut if $\Delta = 0$ at $\rho = |L|$. This means:

$$M = L \left(1 + \frac{4}{3} \Lambda^2 \right) \quad (5.4)$$

which is also the condition of self-duality of the Weyl tensor of the metric (4.4) [2, 10]. Thus, the condition of regularity provides precisely the relation between the two parameters (L and M) such that the metric has a (anti-)self-dual Weyl tensor. Assuming this relation (5.4), one finds:

$$\Delta = (\rho - L)^2 - \frac{1}{3} \Lambda (\rho + 3L)(\rho - L)^3. \quad (5.5)$$

It is easy to see that the condition of regularity is automatically fulfilled.

Taub-Bolt-(anti-)de Sitter

If $\Delta = 0$ has a root at $\rho \neq |L|$, then the set of fixed points of $\partial/\partial\psi$ is necessarily a two dimensional bolt. If the bolt is at ρ_{bolt} , one has:

$$M = \frac{1}{6} \frac{3 \rho_{bolt}^2 - \Lambda \rho_{bolt}^4 + 3 L^2 + 3 \Lambda L^4 + 6 \Lambda L^2 \rho_{bolt}^2}{\rho_{bolt}} \quad (5.6)$$

The condition of regularity then reads:

$$\frac{-\Lambda \rho_{bolt}^2 + L^2 \Lambda + 1}{\rho_{bolt}} = \frac{1}{2L}, \quad (5.7)$$

which requires $L < \rho_{bolt} < 2L$ in the case of positive cosmological constant and $\rho_{bolt} > 2L$ for negative cosmological constant, since $\rho > L$ for L positive. Eq.(5.7) can be solved to locate the bolt which is at

$$\rho_{bolt} = \frac{1}{4} \frac{-1 + \sqrt{1 + 16 \Lambda^2 L^4 + 16 L^2 \Lambda}}{\Lambda L} \quad (5.8)$$

for positive cosmological constant, whence

$$M = \frac{1}{96} \frac{1 + \sqrt{1 + 16 \Lambda^2 L^4 + 16 \Lambda L^2} (8 \Lambda L^2 + 32 \Lambda^2 L^4 - 1)}{\Lambda^2 L^3}. \quad (5.9)$$

However, when the cosmological constant is negative (written as $\Lambda \equiv -\lambda$), the bolt would be either at

$$\rho_{bolt} = \frac{1}{4} \frac{1 - \sqrt{1 + 16 \lambda^2 L^4 - 16 L^2 \lambda}}{\lambda L} \quad (5.10)$$

or at

$$\rho_{bolt} = \frac{1}{4} \frac{1 + \sqrt{1 + 16 \lambda^2 L^4 - 16 L^2 \lambda}}{\lambda L} \quad (5.11)$$

provided that the quantity under the square root is non-negative. This last requirement, together with that of $\rho_{bolt} > 2L$, restricts L :

$$\lambda L^2 \leq \left(\frac{1}{2} - \frac{\sqrt{3}}{4} \right) \quad (\sim 0.066987298) \quad (5.12)$$

Therefore *only* for this range of L can one get a regular bolt, and M is:

$$M = \frac{1}{96} \frac{1 \pm \sqrt{1 + 16 \lambda^2 L^4 - 16 \lambda L^2} (32 \lambda^2 L^4 - 8 \lambda L^2 - 1)}{\lambda^2 L^3}. \quad (5.13)$$

The positive and negative signs correspond to the first and second values of ρ_{bolt} above, respectively. Hence, for an L which is in the permissible range, there are two choices which give a regular bolt, depending on the choice of M .

5.2 The Nuts and Bolts of the new Solutions

Having recalled the above properties of the Taub-Nut/Bolt-(anti-)de Sitter family of metrics, we can now analyze our solutions systematically. The condition for regularity for the metric at the bolt in this case is:

$$\left(1 - \frac{3}{\mu} \right) \frac{d}{d\zeta} \left(\frac{\tilde{\Delta}}{\zeta^2 - l^2} \right)_{(\zeta=\zeta_{root}=l)} = \frac{1}{2l}. \quad (5.14)$$

As before this works as a necessary condition for regularity near a nut. However, near the nut one requires the metric to approach the flat metric on \mathbb{E}^4 which is not guaranteed *a priori* by it.

No Regular Nuts

It is fairly straightforward to see that there will be no regular nut solutions. For a nut, $\tilde{\Delta}$ will have a root at $\zeta = l$, which implies that

$$\tilde{\Delta} = (\zeta - l)^2 - \frac{1}{3} \frac{\Lambda}{(1 - \frac{3}{\mu})} (\zeta + 3l)(\zeta - l)^3. \quad (5.15)$$

It is easy to check that the only way to satisfy condition (5.14) is by taking μ to infinity. Note that, as $\zeta \rightarrow l$, the terms involving Λ in $\tilde{\Delta}$ fall faster than the other terms not involving Λ . Hence, near $\zeta = l$, the metric is the matter-equivalent to the self-dual Taub-Nut instanton [12]:

$$ds^2 = \frac{1}{(1 - \frac{3}{\mu})} \left(\frac{\zeta + l}{\zeta - l} \right) d\zeta^2 + 4(1 - \frac{3}{\mu})l^2 \left(\frac{\zeta - l}{\zeta + l} \right) (d\psi + \cos \theta d\phi)^2 + (\zeta^2 - l^2)(d\theta^2 + \sin^2 \theta d\phi^2) \quad (5.16)$$

When the cosmological constant is zero, this is the metric away from the nut as well. In any case it is straightforward to check directly that the metric (5.16) can not approach flat metric near the nut. Thus, it is not possible to have a regular nut solution of this type. However, the situation with bolts is different.

The Regular Bolts

For the following computations, we set $s = (1 - \frac{3}{\mu})$. The metric then becomes:

$$ds^2 = \frac{\zeta^2 - l^2}{\tilde{\Delta} s} d\zeta^2 + \frac{4l^2 \tilde{\Delta} s}{\zeta^2 - l^2} (d\psi + \cos \theta d\phi)^2 + (\zeta^2 - l^2)(d\theta^2 + \sin^2 \theta d\phi^2), \quad (5.17)$$

where

$$\tilde{\Delta} = \zeta^2 - 2m\zeta + l^2 + \frac{\Lambda}{s} (l^4 + 2l^2\zeta^2 - \frac{1}{3}\zeta^4). \quad (5.18)$$

The regularity condition reads:

$$s \frac{d}{d\zeta} \left(\frac{\tilde{\Delta}}{\zeta^2 - l^2} \right)_{(\zeta=\zeta_{bolt})} = \frac{1}{2l}, \quad (5.19)$$

which for positive cosmological constant reads:

$$2\Lambda l \zeta_{bolt}^2 + \zeta_{bolt} - 2\Lambda l^3 - 2sl = 0, \quad (5.20)$$

and hence requires $l < \zeta_{bolt} < 2sl$. This shows that one must have $s > 1/2$ to have any regular bolt solution (as $\zeta \geq l$) in the case of a positive cosmological constant. Note that, in the limiting case $s = 1/2$, it is *not* possible to have a regular bolt. However, when $s > 1/2$, the bolt is at:

$$\zeta_{bolt} = \frac{1}{4} \frac{-1 + \sqrt{1 + 16\Lambda^2 l^4 + 16l^2 \Lambda s}}{\Lambda l}. \quad (5.21)$$

In the case of a negative cosmological constant, $\Lambda \equiv -\lambda$, the situation becomes much more interesting. The regularity condition then reads:

$$\frac{\lambda \zeta_{bolt}^2 - l^2 \lambda + s}{\zeta_{bolt}} = \frac{1}{2l}, \quad (5.22)$$

which will require $\zeta_{bolt} > 2ls$ in this case, since $\zeta > l$. This limits s to be greater than or equal to $1/2$. The bolts are located at

$$\zeta_{bolt} = \frac{1}{4} \frac{1 - \sqrt{1 + 16\lambda^2 l^4 - 16\lambda l^2 s}}{\lambda l} \quad (5.23)$$

or

$$\zeta_{bolt} = \frac{1}{4} \frac{1 + \sqrt{1 + 16 \lambda^2 l^4 - 16 \lambda l^2 s}}{\lambda l}. \quad (5.24)$$

These correspond to the two different “choices” of m and hence would not appear together in the same metric.

The quantity under the square root in Eqs.(5.23) and (5.24) should be non-negative. Together with the restriction $\zeta_{bolt} > 2l s$, this puts a limit to the range of l :

$$\lambda l^2 \leq \frac{1}{2} s - \frac{1}{4} \sqrt{(4s^2 - 1)}. \quad (5.25)$$

For the limiting case $s = 1/2$, there can hence only be one regular bolt when the cosmological constant is negative. In this case the bolt is located at:

$$\zeta_{bolt} = \frac{1}{4} \frac{2 - 4l^2 \lambda}{\lambda l} \quad (5.26)$$

which implies the restriction:

$$l^2 \lambda < \frac{1}{4} \quad (5.27)$$

(The other possibility is discarded as it places the bolt at $\zeta = l$.) One can now calculate m to be:

$$m = \frac{1}{24} \frac{64 l^6 \lambda^3 - 24 \lambda^2 l^4 + 1}{l^3 \lambda^2} \quad (5.28)$$

Therefore the one-parameter family of metrics satisfying the Einstein equations with a negative cosmological constant and scalar field on a “unit” $\mathbb{C}P^2$ with scalar self-coupling equal to $\frac{1}{2}$ is:

$$ds^2 = \frac{\zeta^2 - l^2}{F} d\zeta^2 + \frac{4l^2 F}{\zeta^2 - l^2} (d\psi + \cos \theta d\phi)^2 + (\zeta^2 - l^2) (d\theta^2 + \sin^2 \theta d\phi^2), \quad (5.29)$$

where

$$F = \frac{1}{24} \frac{(2 \lambda l \zeta + 2 \lambda l^2 - 1) (4 l^2 \lambda^2 \zeta^3 - 4 \zeta^2 l^3 \lambda^2 + 2 l \lambda \zeta^2 - 20 \zeta \lambda^2 l^4 + 2 \lambda l^2 \zeta + \zeta - 12 l^5 \lambda^2)}{l^3 \lambda^2}. \quad (5.30)$$

ζ starts from $\frac{1}{4} \frac{2 - 4l^2 \lambda}{\lambda l}$ and goes to infinity; it is regular everywhere provided that $l^2 \lambda < \frac{1}{4}$. For values of $s > \frac{1}{2}$, the two solutions are found by substituting the two values of m in (5.18).

5.3 $\mathbb{C}P^1$ sigma-model

The scalar manifold of $\mathbb{C}P^1$ has one complex scalar coordinate, a , and the metric

$$ds_{\mathbb{C}P^1}^2 = \frac{da da^*}{(1 + aa^*)^2}. \quad (5.31)$$

By the coordinate transformation

$$a = \tan \left(\frac{\Theta}{2} \right) \exp^{i\Phi}, \quad (5.32)$$

where $0 \leq \Theta \leq \pi$ and $0 \leq \Phi \leq 2\pi$, $\mathbb{C}P^1$ can be given the real metric

$$ds_{\mathbb{C}P^1}^2 = \frac{1}{4} (d\Theta^2 + \sin^2 \Theta d\Phi^2). \quad (5.33)$$

This is the standard round metric on S^2 which satisfies the Einstein equation with a cosmological constant of $1/4$. Note that, as in the case of the Fubini-Study metric (3.5) on \mathbb{CP}^2 , the metric (5.31) is defined up to a positive arbitrary constant. In the case of \mathbb{CP}^2 , the Fubini-Study metric on \mathbb{CP}^2 collapses to a “bolt” (S^2 of constant radius) with the same metric as (5.33) at $R = \infty$ (cf (3.5)). Therefore, the $R = \infty$ solutions for \mathbb{CP}^2 can all be obtained from the \mathbb{CP}^1 case by taking scalar Ansatz $\Phi = \phi, \Theta = \theta$.

6 Conclusion

In this paper we have found some Euclidean solutions to the Einstein field equations when \mathbb{CP}^1 and \mathbb{CP}^2 -sigma-models are coupled to gravity. Such manifolds arise as scalar manifolds in supergravity plus supermatter Lagrangians. In the case of \mathbb{CP}^2 , the sigma-model treatment is linked to the symmetry of spacetime, here given by the group $SU(2) \times U(1)$. Among the solutions obtained in this paper, there is a special class which are in 1-1 correspondence with the two-parameter Taub-NUT-(anti-)de Sitter family of metrics, and can be put in closed form. They exist as a result of the non-trivial topology of \mathbb{CP}^2 and can have bolt-regularity – however, the “distortion” made by the presence of matter prohibits the possibility of any nut-type regularity. Finally, we have shown how these metrics are also obtainable by coupling \mathbb{CP}^1 to gravity.

All solutions and much of the analysis in this paper goes through equally to the Lorentzian régime if one takes an Ansatz of the form $R = R(t)$. The analogous special-case solutions are readily obtainable by taking $r \rightarrow it$ and are all in 1-1 correspondence with Lorentzian Taub-NUT-(anti-)de Sitter spacetimes. The dynamics of the general field equations, i.e., the ‘ $r \rightarrow it$ ’-version of Eq.(3.13)-(3.14), is left for future investigation.

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